



# Variational approach to solutions for a class of fractional boundary value problems

Ziheng Zhang<sup>✉1</sup> and Jing Li<sup>2</sup>

<sup>1</sup>Department of Mathematics, Tianjin Polytechnic University, Tianjin 300387, China

<sup>2</sup>School of Management, Qingdao Huanghai University, Qingdao 266427, Shandong, China

Received 11 April 2014, appeared 12 March 2015

Communicated by Gabriele Bonanno

**Abstract.** In this paper we investigate the existence of infinitely many solutions for the following fractional boundary value problem

$$\begin{cases} {}_t D_T^\alpha ({}_0 D_t^\alpha u(t)) = \nabla W(t, u(t)), & t \in [0, T], \\ u(0) = u(T) = 0, \end{cases} \quad (\text{FBVP})$$

where  $\alpha \in (1/2, 1)$ ,  $u \in \mathbb{R}^n$ ,  $W \in C^1([0, T] \times \mathbb{R}^n, \mathbb{R})$  and  $\nabla W(t, u)$  is the gradient of  $W(t, u)$  at  $u$ . The novelty of this paper is that, assuming  $W(t, u)$  is of subquadratic growth as  $|u| \rightarrow +\infty$ , we show that (FBVP) possesses infinitely many solutions via the genus properties in the critical theory. Recent results in the literature are generalized and significantly improved.

**Keywords:** fractional Hamiltonian systems, critical point, variational methods, genus.

**2010 Mathematics Subject Classification:** 34C37, 35A15, 35B38.

## 1 Introduction

Fractional differential equations, both ordinary and partial ones, are extensively applied in mathematical modeling of processes in physics, mechanics, control theory, biochemistry, bio-engineering and economics. Therefore, the theory of fractional differential equations is an area intensively developed during the last decades [4]. The monographs [7, 9, 12] enclose a review of methods of solving fractional differential equations.

Recently, equations including both left and right fractional derivatives are discussed. Apart from their possible applications, equations with left and right derivatives provide an interesting and new field in fractional differential equations theory. In this topic, many results are obtained dealing with the existence and multiplicity of solutions of nonlinear fractional differential equations by using techniques of nonlinear analysis, such as fixed point theory (including Leray–Schauder nonlinear alternative), topological degree theory (including co-incidence degree theory) and comparison method (including upper and lower solutions and monotone iterative method).

<sup>✉</sup>Corresponding author. Email: [zhzh@mail.bnu.edu.cn](mailto:zhzh@mail.bnu.edu.cn)

It should be noted that critical point theory and variational methods have also turned out to be very effective tools in determining the existence of solutions for integer order differential equations. The idea behind them is trying to find solutions of a given boundary value problem by looking for critical points of a suitable energy functional defined on an appropriate function space. In the last 30 years, the critical point theory has become a wonderful tool in studying the existence of solutions to differential equations with variational structures, we refer the reader to the books by Mawhin and Willem [8], Rabinowitz [13], Schechter [16] and the references listed therein.

Motivated by the above classical works, in the recent paper [6], for the first time, the authors showed that critical point theory is an effective approach to tackle the existence of solutions for the following fractional boundary value problem

$$\begin{cases} {}_tD_T^\alpha({}_0D_t^\alpha u(t)) = \nabla W(t, u(t)), & t \in [0, T], \\ u(0) = u(T), \end{cases} \quad (\text{FBVP})$$

where  $\alpha \in (1/2, 1)$ ,  $u \in \mathbb{R}^n$ ,  $W \in C^1([0, T] \times \mathbb{R}^n, \mathbb{R})$  and  $\nabla W(t, u)$  is the gradient of  $W(t, u)$  at  $u$ . Explicitly, under the assumption that

$$(H_1) \quad |W(t, u)| \leq \bar{a}|u|^2 + \bar{b}(t)|u|^{2-\tau} + \bar{c}(t) \text{ for all } t \in [0, T] \text{ and } u \in \mathbb{R}^n,$$

where  $\bar{a} \in [0, \Gamma^2(\alpha + 1)/2T^{2\alpha})$ ,  $\tau \in (0, 2)$ ,  $\bar{b} \in L^{2/\tau}[0, T]$  and  $\bar{c} \in L^1[0, T]$ , combining with some other reasonable hypotheses on  $W(t, u)$ , the authors showed that (FBVP) has at least one nontrivial solution. In addition, assuming that the potential  $W(t, u)$  satisfies the following superquadratic condition:

$$(H_2) \quad \text{there exists } \mu > 2 \text{ and } R > 0 \text{ such that}$$

$$0 < \mu W(t, u) \leq (\nabla W(t, u), u)$$

for all  $t \in [0, T]$  and  $u \in \mathbb{R}^n$  with  $|u| \geq R$ ,

and some other assumptions on  $W(t, u)$ , they also obtained the existence of at least one nontrivial solution for (FBVP). Inspired by this work, in [18] the author considered the following fractional boundary value problem

$$\begin{cases} {}_tD_T^\alpha({}_0D_t^\alpha u(t)) = f(t, u(t)), & t \in [0, T], \\ u(0) = u(T) = 0, \end{cases} \quad (1.1)$$

with  $\alpha \in (1/2, 1)$ ,  $u \in \mathbb{R}$ ,  $f: [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$  satisfying the following hypotheses:

$$(f_1) \quad f \in C([0, T] \times \mathbb{R}, \mathbb{R});$$

$$(f_2) \quad \text{there is a constant } \mu > 2 \text{ such that}$$

$$0 < \mu F(t, u) \leq u f(t, u) \quad \text{for all } t \in [0, T] \text{ and } u \in \mathbb{R} \setminus \{0\},$$

the author showed that (1.1) possesses at least one nontrivial solution via the mountain pass theorem. For the other works related to the solutions of fractional boundary value problems, we refer the reader to the papers [2, 5, 10, 17, 19] and the references mentioned there.

Note that all the papers mentioned above are concerned with the existence of solutions for (FBVP). As far as the multiplicity of solutions for (FBVP) is concerned, to the best of

our knowledge, there is no result about this. Therefore, motivated by the above results, in this paper using the genus properties of critical point theory we establish some new criterion to guarantee the existence of infinitely many solutions of (FBVP) for the case that  $W(t, u)$  is subquadratic as  $|u| \rightarrow +\infty$ . Note that in [11] the same techniques are used to consider the existence of solutions to nonlocal Kirchhoff equations of elliptic type. For the statement of our main result in the present paper, the potential  $W(t, u)$  is supposed to satisfy the following conditions:

(W<sub>1</sub>)  $W(t, 0) = 0$  for all  $t \in [0, T]$ ,  $W(t, u) \geq a(t)|u|^\vartheta$  and  $|\nabla W(t, u)| \leq b(t)|u|^{\vartheta-1}$  for all  $(t, u) \in [0, T] \times \mathbb{R}^n$ , where  $1 < \vartheta < 2$  is a constant,  $a: [0, T] \rightarrow \mathbb{R}^+$  is a continuous function and  $b: [0, T] \rightarrow \mathbb{R}^+$  is a continuous function;

(W<sub>2</sub>) there is a constant  $1 < \sigma \leq \vartheta < 2$  such that

$$(\nabla W(t, u), u) \leq \sigma W(t, u) \quad \text{for all } t \in [0, T] \text{ and } u \in \mathbb{R}^n.$$

Now, we can state our main result.

**Theorem 1.1.** *Suppose that (W<sub>1</sub>) and (W<sub>2</sub>) are satisfied. Moreover, assume that  $W(t, u)$  is even in  $u$ , i.e.,*

(W<sub>3</sub>)  $W(t, u) = W(t, -u)$  for all  $t \in [0, T]$  and  $u \in \mathbb{R}^n$ ,

*then (FBVP) has infinitely many nontrivial solutions.*

**Remark 1.2.** From (W<sub>1</sub>), it is easy to check that  $W(t, u)$  is subquadratic as  $|u| \rightarrow +\infty$ . In fact, in view of (W<sub>1</sub>), we have

$$W(t, u) = \int_0^1 (\nabla W(t, su), u) ds \leq \frac{b(t)}{\vartheta} |u|^\vartheta, \quad (1.2)$$

which implies that  $W(t, u)$  is of subquadratic growth as  $|u| \rightarrow +\infty$ .

(H<sub>2</sub>) is the so-called Ambrosetti–Rabinowitz condition due to Ambrosetti and Rabinowitz (see e.g., [1]), which implies that  $W(t, u)$  is superquadratic as  $|u| \rightarrow +\infty$ . Here we consider the case that  $W(t, u)$  is of subquadratic growth. Therefore, the result in [18] is complemented. In addition, in view of (1.2), it is obvious that if  $W(t, u)$  satisfies (W<sub>1</sub>), then (H<sub>1</sub>) holds. However, in [6] the authors only obtained the existence of at least one nontrivial solution for (FBVP). In our Theorem 1.1, we obtain that (FBVP) possesses infinitely many nontrivial solutions.

**Example 1.3.** Here we give an example to illustrate Theorem 1.1. Take

$$W(t, u) = (2 + \sin t)|u|^{\frac{3}{2}}, \quad \forall (t, u) \in [0, T] \times \mathbb{R},$$

then it is easy to check that (W<sub>1</sub>), (W<sub>2</sub>) and (W<sub>3</sub>) are satisfied where  $a(t) = 2 + \sin t$ ,  $b(t) = \frac{3}{2}(2 + \sin t)$  and  $\sigma = \vartheta = \frac{3}{2}$ .

The remaining part of this paper is organized as follows. Some preliminary results are presented in Section 2. In Section 3, we are devoted to accomplishing the proof of Theorem 1.1.

## 2 Preliminary results

In this section, for the reader's convenience, firstly we introduce some basic definitions of fractional calculus which are used further in this paper, see [7].

**Definition 2.1** (Left and right Riemann–Liouville fractional integrals). Let  $u$  be a function defined on  $[a, b]$ . The left and right Riemann–Liouville fractional integrals of order  $\alpha > 0$  for function  $u$  are defined by

$${}_a I_t^\alpha u(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} u(s) ds, \quad t \in [a, b]$$

and

$${}_t I_b^\alpha u(t) = \frac{1}{\Gamma(\alpha)} \int_t^b (s-t)^{\alpha-1} u(s) ds, \quad t \in [a, b].$$

**Definition 2.2** (Left and right Riemann–Liouville fractional derivatives). Let  $u$  be a function defined on  $[a, b]$ . The left and right Riemann–Liouville fractional derivatives of order  $\alpha > 0$  for function  $u$  denoted by  ${}_a D_t^\alpha u(t)$  and  ${}_t D_b^\alpha u(t)$ , respectively, are defined by

$${}_a D_t^\alpha u(t) = \frac{d^n}{dt^n} {}_a T_t^{n-\alpha} u(t)$$

and

$${}_t D_b^\alpha u(t) = (-1)^n \frac{d^n}{dt^n} {}_t I_b^{n-\alpha} u(t),$$

where  $t \in [a, b]$ ,  $n-1 \leq \alpha < n$  and  $n \in \mathbb{N}$ .

In what follows, to establish the variational structure which enables us to reduce the existence of solutions for (FBVP) to find critical points of the corresponding functional, it is necessary to construct appropriate function spaces.

We recall some fractional spaces, for more details see [3]. To this end, denote by  $L^p[0, T]$  ( $1 < p < +\infty$ ) the Banach spaces of functions on  $[0, T]$  with values in  $\mathbb{R}^n$  under the norms

$$\|u\|_p = \left( \int_0^T |u(t)|^p dt \right)^{1/p},$$

and  $L^\infty[0, T]$  is the Banach space of essentially bounded functions from  $[0, T]$  into  $\mathbb{R}^n$  equipped with the norm

$$\|u\|_\infty = \text{ess sup } \{|u(t)| : t \in [0, T]\}.$$

For  $0 < \alpha \leq 1$  and  $1 < p < +\infty$ , the fractional derivative space  $E_0^{\alpha, p}$  is defined by

$$E_0^{\alpha, p} = \{u \in L^p[0, T] : {}_0 D_t^\alpha u \in L^p[0, T] \text{ and } u(0) = u(T) = 0\} = \overline{C_0^\infty[0, T]}^{\|\cdot\|_{\alpha, p}},$$

where  $\|\cdot\|_{\alpha, p}$  is defined as follows

$$\|u\|_{\alpha, p} = \left( \int_0^T |u(t)|^p dt + \int_0^T |{}_0 D_t^\alpha u(t)|^p dt \right)^{1/p}. \quad (2.1)$$

Then  $E_0^{\alpha, p}$  is a reflexive and separable Banach space.

**Lemma 2.3** ([6, Proposition 3.3]). Let  $0 < \alpha \leq 1$  and  $1 < p < +\infty$ . For all  $u \in E_0^{\alpha,p}$ , if  $\alpha > \frac{1}{p}$ , we have

$${}_0I_t^\alpha({}_0D_t^\alpha u(t)) = u(t)$$

and

$$\|u\|_p \leq \frac{T^\alpha}{\Gamma(\alpha+1)} \|{}_0D_t^\alpha u\|_p. \quad (2.2)$$

In addition, if  $\alpha > \frac{1}{p}$  and  $\frac{1}{p} + \frac{1}{q} = 1$ , then

$$\|u\|_\infty \leq \frac{T^{\alpha-\frac{1}{p}}}{\Gamma(\alpha)((\alpha-1)q+1)^{\frac{1}{q}}} \|{}_0D_t^\alpha u\|_p.$$

**Remark 2.4.** According to (2.2), we can consider in  $E_0^{\alpha,p}$  the following norm

$$\|u\|_{\alpha,p} = \|{}_0D_t^\alpha u\|_p, \quad (2.3)$$

which is equivalent to (2.1).

In what follows we denote by  $E^\alpha = E_0^{\alpha,2}$ . Then it is a Hilbert space with respect to the norm  $\|u\|_\alpha = \|u\|_{\alpha,2}$  given by (2.3).

The main difficulty in dealing with the existence of infinitely many solutions for (FBVP) is to verify that the functional corresponding to (FBVP) satisfies (PS)-condition. To overcome this difficulty, we need the following proposition.

**Proposition 2.5** ([6, Proposition 3.4]). Let  $0 < \alpha \leq 1$  and  $1 < p < +\infty$ . Assume that  $\alpha > \frac{1}{p}$  and  $u_k \rightharpoonup u$  in  $E_0^{\alpha,p}$ , then  $u_k \rightarrow u$  in  $C[0, T]$ , i.e.,

$$\|u_k - u\|_\infty \rightarrow 0$$

as  $k \rightarrow +\infty$ .

Now we introduce more notations and some necessary definitions. Let  $\mathcal{B}$  be a real Banach space,  $I \in C^1(\mathcal{B}, \mathbb{R})$  means that  $I$  is a continuously Fréchet differentiable functional defined on  $\mathcal{B}$ .

**Definition 2.6.**  $I \in C^1(\mathcal{B}, \mathbb{R})$  is said to satisfy (PS)-condition if any sequence  $\{u_k\}_{k \in \mathbb{N}} \subset \mathcal{B}$ , for which  $\{I(u_k)\}_{k \in \mathbb{N}}$  is bounded and  $I'(u_k) \rightarrow 0$  as  $k \rightarrow +\infty$ , possesses a convergent subsequence in  $\mathcal{B}$ .

In order to find infinitely many solutions of (FBVP) under the assumptions of Theorem 1.1, we shall use the “genus” properties. Therefore, it is necessary to recall the following definitions and results, see [13, 14].

Let  $\mathcal{B}$  be a Banach space,  $I \in C^1(\mathcal{B}, \mathbb{R})$  and  $c \in \mathbb{R}$ . We set

$$\begin{aligned} \Sigma &= \{A \subset \mathcal{B} - \{0\} : A \text{ is closed in } \mathcal{B} \text{ and symmetric with respect to } 0\}, \\ K_c &= \{u \in \mathcal{B} : I(u) = c, I'(u) = 0\}, \quad I^c = \{u \in \mathcal{B} : I(u) \leq c\}. \end{aligned}$$

**Definition 2.7.** For  $A \in \Sigma$ , we say the genus of  $A$  is  $j$  (denoted by  $\gamma(A) = j$ ) if there is an odd map  $\psi \in C(A, \mathbb{R}^j \setminus \{0\})$  and  $j$  is the smallest integer with this property.

**Lemma 2.8.** *Let  $I$  be an even  $C^1$  functional on  $\mathcal{B}$  and satisfy (PS)-condition. For any  $j \in \mathbb{N}$ , set*

$$\Sigma_j = \{A \in \Sigma : \gamma(A) \geq j\}, \quad c_j = \inf_{A \in \Sigma_j} \sup_{u \in A} I(u).$$

(i) *If  $\Sigma_j \neq \emptyset$  and  $c_j \in \mathbb{R}$ , then  $c_j$  is a critical value of  $I$ ;*

(ii) *if there exists  $r \in \mathbb{N}$  such that*

$$c_j = c_{j+1} = \cdots = c_{j+r} = c \in \mathbb{R},$$

*and  $c \neq I(0)$ , then  $\gamma(K_c) \geq r + 1$ .*

**Remark 2.9.** From Remark 7.3 in [13], we know that if  $K_c \subset \Sigma$  and  $\gamma(K_c) > 1$ , then  $K_c$  contains infinitely many distinct points, i.e.,  $I$  has infinitely many distinct critical points in  $\mathcal{B}$ .

### 3 Proof of Theorem 1.1

The aim of this section is to give the proof of Theorem 1.1. To do this, we are going to establish the corresponding variational framework of (FBVP). Define the functional  $I: \mathcal{B} = E^\alpha \rightarrow \mathbb{R}$  by

$$I(u) = \int_0^T \left[ \frac{1}{2} |{}_0D_t^\alpha u(t)|^2 - W(t, u(t)) \right] dt. \quad (3.1)$$

**Lemma 3.1** ([6, Corollary 3.1]). *Under the conditions of Theorem 1.1,  $I$  is a continuously Fréchet-differentiable functional defined on  $E^\alpha$ , i.e.,  $I \in C^1(E^\alpha, \mathbb{R})$ . Moreover, we have*

$$I'(u)v = \int_0^T [({}_0D_t^\alpha u(t), {}_0D_t^\alpha v(t)) - (\nabla W(t, u(t)), v(t))] dt$$

for all  $u, v \in E^\alpha$ , which yields that

$$I'(u)u = \int_0^T |{}_0D_t^\alpha u(t)|^2 dt - \int_0^T (\nabla W(t, u(t)), u(t)) dt. \quad (3.2)$$

**Lemma 3.2.** *If  $(W_1)$  and  $(W_2)$  hold, then  $I$  satisfies (PS)-condition.*

*Proof.* Assume that  $\{u_k\}_{k \in \mathbb{N}} \subset E^\alpha$  is a sequence such that  $\{I(u_k)\}_{k \in \mathbb{N}}$  is bounded and  $I'(u_k) \rightarrow 0$  as  $k \rightarrow +\infty$ . Then there exists a constant  $M > 0$  such that

$$|I(u_k)| \leq M \quad \text{and} \quad \|I'(u_k)\|_{(E^\alpha)^*} \leq M \quad (3.3)$$

for every  $k \in \mathbb{N}$ , where  $(E^\alpha)^*$  is the dual space of  $E^\alpha$ .

We firstly prove that  $\{u_k\}_{k \in \mathbb{N}}$  is bounded in  $E^\alpha$ . From (3.1) and (3.2), we obtain that

$$\begin{aligned} \left(1 - \frac{\sigma}{2}\right) \|u_k\|_\alpha^2 &= I'(u_k)u_k - \sigma I(u_k) \\ &+ \int_0^T [(\nabla W(t, u_k(t)), u_k(t)) - \sigma W(t, u_k(t))] dt \\ &\leq M \|u_k\|_\alpha + \sigma M. \end{aligned} \quad (3.4)$$

Since  $1 < \sigma < 2$ , the inequality (3.4) shows that  $\{u_k\}_{k \in \mathbb{N}}$  is bounded in  $E^\alpha$ . Then the sequence  $\{u_k\}_{k \in \mathbb{N}}$  has a subsequence, again denoted by  $\{u_k\}_{k \in \mathbb{N}}$ , and there exists  $u \in E^\alpha$  such that

$$u_k \rightharpoonup u \text{ weakly in } E^\alpha,$$

which yields that

$$(I'(u_k) - I'(u))(u_k - u) \rightarrow 0. \quad (3.5)$$

Moreover, according to Proposition 2.5, we have

$$\int_0^T (\nabla W(t, u_k(t)) - \nabla W(t, u(t)), u_k(t) - u(t)) dt \rightarrow 0 \quad (3.6)$$

as  $k \rightarrow +\infty$ . Consequently, combining (3.5), (3.6) with the following equality

$$(I'(u_k) - I'(u))(u_k - u) = \|u_k - u\|_\alpha^2 - \int_0^T (\nabla W(t, u_k(t)) - \nabla W(t, u(t)), u_k(t) - u(t)) dt,$$

we deduce that  $\|u_k - u\|_\alpha \rightarrow 0$  as  $k \rightarrow +\infty$ . That is,  $I$  satisfies the (PS)-condition.  $\square$

Now we are in the position to complete the proof of Theorem 1.1.

*Proof of Theorem 1.1.* According to  $(W_1)$  and  $(W_3)$ , it is obvious that  $I$  is even and  $I(0) = 0$ . In order to apply Lemma 2.8, we prove that

$$\text{for any } j \in \mathbb{N} \text{ there exists } \varepsilon > 0 \text{ such that } \gamma(I^{-\varepsilon}) \geq j. \quad (3.7)$$

To do this, let  $\{e_j\}_{j=1}^\infty$  be the standard orthogonal basis of  $E^\alpha$ , i.e.,

$$\|e_i\|_\alpha = 1 \quad \text{and} \quad \langle e_i, e_k \rangle_{E^\alpha} = 0, \quad 1 \leq i \neq k. \quad (3.8)$$

For any  $j \in \mathbb{N}$ , define

$$E_j^\alpha = \text{span}\{e_1, e_2, \dots, e_j\}, \quad S_j = \{u \in E_j^\alpha : \|u\|_\alpha = 1\},$$

then, for any  $u \in E_j^\alpha$ , there exist  $\lambda_i \in \mathbb{R}$ ,  $i = 1, 2, \dots, j$ , such that

$$u(t) = \sum_{i=1}^j \lambda_i e_i(t) \quad \text{for } t \in [0, T], \quad (3.9)$$

which indicates that

$$\begin{aligned} \|u\|_\alpha^2 &= \int_0^T |{}_0D_t^\alpha u(t)|^2 dt \\ &= \sum_{i=1}^j \lambda_i^2 \int_0^T |{}_0D_t^\alpha e_i(t)|^2 dt \\ &= \sum_{i=1}^j \lambda_i^2 \|e_i\|_\alpha^2 = \sum_{i=1}^j \lambda_i^2. \end{aligned} \quad (3.10)$$

On the other hand, in view of  $(W_1)$ , for any bounded open set  $D \subset [0, T]$ , there exists  $\eta > 0$  (dependent on  $D$ ) such that

$$W(t, u) \geq a(t)|u|^\vartheta \geq \eta|u|^\vartheta, \quad (t, u) \in D \times \mathbb{R}^n. \quad (3.11)$$

As a result, for any  $u \in S_j$ , we can take some  $D_0 \subset [0, T]$  such that

$$\int_0^T W(t, u(t)) dt = \int_0^T W\left(t, \sum_{i=1}^j \lambda_i e_i(t)\right) dt \geq \eta \int_{D_0} \left| \sum_{i=1}^j \lambda_i e_i(t) \right|^\vartheta dt =: \varrho > 0. \quad (3.12)$$

Indeed, if not, for any bounded open set  $D \subset [0, T]$ , there exists  $\{u_n\}_{n \in \mathbb{N}} \in S_j$  such that

$$\int_D |u_n(t)|^\theta dt = \int_D \left| \sum_{i=1}^j \lambda_{in} e_i(t) \right|^\theta dt \rightarrow 0$$

as  $n \rightarrow +\infty$ , where  $u_n = \sum_{i=1}^j \lambda_{in} e_i$  such that  $\sum_{i=1}^j \lambda_{in}^2 = 1$ . Since  $\sum_{i=1}^j \lambda_{in}^2 = 1$ , we have

$$\lim_{n \rightarrow +\infty} \lambda_{in} =: \lambda_{i0} \in [-1, 1] \quad \text{and} \quad \sum_{i=1}^j \lambda_{i0}^2 = 1.$$

Hence, for any bounded open set  $D \subset [0, T]$ , it follows that

$$\int_D \left| \sum_{i=1}^j \lambda_{i0} e_i(t) \right|^\theta dt = 0.$$

The fact that  $D$  is arbitrary yields that  $u_0 = \sum_{i=1}^j \lambda_{i0} e_i(t) = 0$  a.e. on  $[0, T]$ , which contradicts the fact that  $\|u_0\|_\alpha = 1$ . Hence, (3.12) holds.

Consequently, according to  $(W_1)$  and (3.9)–(3.12), we have

$$\begin{aligned} I(su) &= \frac{s^2}{2} \|u\|_\alpha^2 - \int_0^T W(t, su(t)) dt \\ &= \frac{s^2}{2} \|u\|_\alpha^2 - \int_0^T W\left(t, s \sum_{i=1}^j \lambda_i e_i(t)\right) dt \\ &\leq \frac{s^2}{2} \|u\|_\alpha^2 - s^\theta \int_0^T a(t) \left| \sum_{i=1}^j \lambda_i e_i(t) \right|^\theta dt \\ &\leq \frac{s^2}{2} \|u\|_\alpha^2 - \eta s^\theta \int_{D_0} \left| \sum_{i=1}^j \lambda_i e_i(t) \right|^\theta dt \\ &\leq \frac{s^2}{2} \|u\|_\alpha^2 - \varrho s^\theta \\ &= \frac{s^2}{2} - \varrho s^\theta, \quad u \in S_j, \end{aligned}$$

which implies that there exist  $\varepsilon > 0$  and  $\delta > 0$  such that

$$I(\delta u) < -\varepsilon \quad \text{for } u \in S_j. \quad (3.13)$$

Let

$$S_j^\delta = \{\delta u : u \in S_j\} \quad \text{and} \quad \Omega = \left\{ (\lambda_1, \lambda_2, \dots, \lambda_j) \in \mathbb{R}^j : \sum_{i=1}^j \lambda_i^2 < \delta^2 \right\}.$$

Then it follows from (3.13) that

$$I(u) < -\varepsilon, \quad \forall u \in S_j^\delta,$$

which, together with the fact that  $I \in C^1(E^\alpha, \mathbb{R})$  is even, yields that

$$S_j^\delta \subset I^{-\varepsilon} \in \Sigma.$$



On the other hand, it follows from (3.9) and (3.10) that there exists an odd homeomorphism mapping  $\psi \in C(S_j^\delta, \partial\Omega)$ . By some properties of the genus (see 3° of Proposition 7.5 and 7.7 in [13]), we obtain

$$\gamma(I^{-\varepsilon}) \geq \gamma(S_j^\delta) = j, \quad (3.14)$$

so (3.7) follows. Set

$$c_j = \inf_{A \in \Sigma_j} \sup_{u \in A} I(u),$$

then, from (3.14) and the fact that  $I$  is bounded from below on  $E^\alpha$ , we have  $-\infty < c_j \leq -\varepsilon < 0$ , that is, for any  $j \in \mathbb{N}$ ,  $c_j$  is a real negative number. By Lemma 2.8 and Remark 2.9,  $I$  has infinitely many nontrivial critical points, and consequently (FBVP) possesses infinitely many nontrivial solutions.  $\square$

## Acknowledgements

The authors would like to express their appreciation to the referee for valuable suggestions. This work is supported by National Natural Science Foundation of China (No. 11101304).

## References

- [1] A. AMBROSETTI, P. H. RABINOWITZ, Dual variational methods in critical point theory and applications, *J. Funct. Anal.* **14**(1973), No. 4, 349–381. [MR0370183](#)
- [2] J. CHEN, X. H. TANG, Existence and multiplicity of solutions for some fractional boundary value problem via critical point theory, *Abstr. Appl. Anal.* **2012**, Art. ID 648635, 21 pp. [MR2872321](#)
- [3] V. J. ERVIN, J. P. ROOP, Variational formulation for the stationary fractional advection dispersion equation, *Numer. Methods Partial Differential Equations* **22**(2006), No. 3, 358–76. [MR2212226](#)
- [4] R. HILFER (Ed.), *Applications of fractional calculus in physics*, World Scientific Publishing Co., Inc., River Edge, NJ, 2000. [MR1890104](#)
- [5] C. Z. HU, B. LIU, S. F. XIE, Monotone iterative solutions for nonlinear boundary value problems of fractional differential equation, *Abstr. Appl. Anal.* **2013**, Art. ID 493164, 8 pp. [MR3108637](#)
- [6] F. JIAO, Y. ZHOU, Existence results for fractional boundary value problem via critical point theory, *Internat. J. Bifur. Chaos Appl. Sci. Engrg.* **22**(2012), No. 4, Art. ID 1250086, 17 pp. [MR2926062](#)
- [7] A. KILBAS, H. SRIVASTAVA, J. TRUJILLO, *Theory and applications of fractional differential equations*, North-Holland Mathematics Studies, Vol. 204, Elsevier Science B.V., Amsterdam, 2006. [MR2218073](#)
- [8] J. MAWHIN, M. WILLEM, *Critical point theory and Hamiltonian systems*, Applied Mathematical Sciences, Vol. 74. Springer-Verlag, New York, 1989. [MR0982267](#)

- [9] K. MILLER, B. ROSS, *An introduction to the fractional calculus and fractional differential equations*, John Wiley & Sons, Inc., New York, 1993. [MR1219954](#)
- [10] N. NYAMORADI, Infinitely many solutions for a class of fractional boundary value problems with Dirichlet boundary conditions, *Mediterr. J. Math.* **11**(2014), No. 1, 75–87. [MR3160613](#)
- [11] N. NYAMORADI, N. T. CHUNG, Existence of solutions to nonlocal Kirchhoff equations of elliptic type via genus theory, *Electron. J. Differential Equations* **2014**, No. 86, 1–12. [MR3193992](#)
- [12] I. PODLUBNY, *Fractional differential equations*, Mathematics in Science and Engineering, Vol. 198, Academic Press, Inc., San Diego, CA, 1999. [MR1658022](#)
- [13] P. H. RABINOWITZ, *Minimax methods in critical point theory with applications to differential equations*, CBMS Regional Conference Series in Mathematics, Vol. 65, Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 1986. [MR0845785](#)
- [14] A. SALVATORE, Homoclinic orbits for a special class of nonautonomous Hamiltonian systems, *Nonlinear Anal.* **30**(1997), No. 8, 4849–4857. [MR1725348](#)
- [15] S. G. SAMKO, A. A. KILBAS, O. I. MARICHEV, *Fractional integrals and derivatives: theory and applications*, Gordon and Breach Science Publishers, Yverdon, 1993. [MR1347689](#)
- [16] M. SCHECHTER, *Linking methods in critical point theory*, Birkhäuser Boston, Inc., Boston, MA, 1999. [MR1729208](#)
- [17] H. R. SUN, Q. G. ZHANG, Existence of solutions for fractional boundary value problem via the Mountain Pass method and an iterative technique, *Comput. Math. Appl.* **64**(2012), 3436–3443. [url](#)
- [18] C. TORRES, Mountain pass solutions for a fractional boundary value problem, *J. Fract. Cal. Appl.* **5**(2014), No. 1, 1–10. [MR3234091](#)
- [19] S. Q. ZHANG, Existence of a solution for the fractional differential equation with nonlinear boundary conditions, *J. Math. Anal. Appl.* **61**(2011), No. 4, 1202–1208. [MR2770522](#)